Bearing Pressures and Cracks

Bearing Pressures Through a Slightly Waved Surface or Through a Nearly Flat Part of a Cylinder, and Related Problems of Cracks

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The task is undertaken of determining the bearing pressures, and the stresses and deformations created by them, in some cases that differ from those considered by Hertz[2] in his classical study of contact. Thus two solids are examined which, before loading, are in contact along a row of evenly spaced lines in a horizontal plane, as indicated in Fig. 1(a). Between these lines the surfaces have a separation defined by a nearly flat cosine wave. A uniform pressure on top of the upper solid creates contact over an area consisting of a row of strips, reduces the separation of the solids between the strips, as suggested in Fig. 1(b), and creates contact pressures distributed as indicated in Fig. 1(c), with vertical rises in the diagram of pressure at the edges of the strips. At a greater load the width of the strip becomes equal to the wave length, and the contact is complete. At still greater loads the stresses increase as if the two solids were one. The procedure by which this problem is solved is demonstrated first by showing its easy application to some well-known cases, especially Hertz's problem of circular cylinders in contact.[2]

Further applications are to a noncircular cylinder resting on a solid with a flat top, with an initial separation of the surfaces varying as the fourth power of the distance from the initial line of contact; to partial contact of two surfaces which are initially plane, except that one of them has a ridge or several parallel ridges; and to some related problems in which two parts of the same body are partially separated by the forming of one or more cracks.

Notation

x, y	= rectangular coordinates, y vertical
r, θ	= corresponding polar coordinates
z	= $x + iy = re^{i\theta}$ = complex variable
Z	= function of z, Equation [1], defining the stresses by Equations [4] to [6]
$Z', \overline{Z}, \overline{\overline{Z}}$	= derivative and first and second integral of Z, Equations [2]
$\sigma_x, \sigma_y, \tau_{xy}$	= normal stresses and shearing stress in the directions of x and y
ξ, η	= displacements in the directions of x and y
η_0	= displacement η at $y = 0$
s	= initial separation of two surfaces
E, G, μ	= Young's modulus, modulus of elasticity in shear, and Poisson's ratio
F	= Airy's stress function
	= force on slice parallel to the x,y-plane one unit thick, measurable in pounds per inch
p	= average pressure or tension, measurable in pounds per square inch
a, l	= horizontal distances on axis of x
c, c_1	= constants

Function of a Complex Variable Used as Stress Function

A stress function will be applied of a type which was introduced by Carothers[3] in 1920 and, evidently independently, by Nádai[4] in 1921. Both expressed the significant values in terms of harmonic functions, and both made use of the following fact: A harmonic function of x and y can be obtained as the real part ReZ or the imaginary part ImZ of an analytic function Z of the complex variable $z = x + iy$, with Z being written in the forms

$$Z = Z(z) = Z(x + iy) = \mathrm{Re}Z + i\mathrm{Im}Z \quad [1]$$

In the present applications it is expedient, as done by MacGregor,[5] to use the function Z itself as stress function.

The further functions Z', $\overline{Z}$, and $\overline{\overline{Z}}$ are the derivative and first and second integrals of Z, so that

$$Z' = \frac{dZ}{dz}, \quad Z = \frac{d\overline{Z}}{dz}, \quad \overline{Z} = \frac{d\overline{\overline{Z}}}{dz} \quad [2]$$

The properties of derivatives are noted

$$\frac{\partial \mathrm{Re}Z}{\partial x} = \frac{\partial \mathrm{Im}Z}{\partial y} = \mathrm{Re}Z', \quad \frac{\partial \mathrm{Im}Z}{\partial x} = -\frac{\partial \mathrm{Re}Z}{\partial y} = \mathrm{Im}Z' \quad [3]$$

In a restricted but important group of cases the normal stresses and the shearing stress in the directions of x and y can be stated in the form

$$\sigma_x = \mathrm{Re}Z - y\mathrm{Im}Z' \quad [4]$$

$$\sigma_y = \mathrm{Re}Z + y\mathrm{Im}Z' \quad [5]$$

$$\tau_{xy} = -y\mathrm{Re}Z' \quad [6]$$

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[2] Heinrich Hertz, Crelle's *Journal für die reine und angewandte Mathematik*, vol. 92, 1881, p. 156 (also in his Gesammelte Werke, vol. 1, 1895, p. 155). See, for example "Theory of Elasticity," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1934, pp. 339–350.

[3] "Plane Strain: The Direct Determination of Stress," by S. D. Carothers, Proceedings of the Royal Society of London, series A, vol. 97, 1920, pp. 110–123, especially p. 119.

[4] "Über die Spannungsverteilung in einer durch eine Einzelkraft belasteten rechteckigen Platte," by A. Nádai, *Der Bauingenieur*, vol. 2, 1921, pp. 11–16, especially p. 12. Nádai applied the function to express curvatures and twists of elastic slabs. The curvatures and twists can be interpreted as stresses through Airy's stress function.

[5] "The Potential Function Method for the Solution of Two-Dimensional Stress Problems," by C. W. MacGregor, Trans. American Mathematical Society, vol. 38, no. 1, July, 1935, pp. 177–186.

By referring to Equations [3] it is observed that these stresses satisfy the two conditions of equilibrium of the form

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \ldots\ldots\ldots\ldots [7]$$

The limitation of this type of solution appears in Equations [4] to [6], which require that

$$\sigma_x = \sigma_y \text{ and } \tau_{xy} = 0 \text{ at } y = 0 \ldots\ldots\ldots\ldots [8]$$

With deformation in the direction perpendicular to the x, y-plane prevented, the displacements ξ and η in the directions of x and y are defined by the formulas

$$2G\xi = (1 - 2\mu)\, \text{Re}\bar{Z} - y\text{Im}Z \ldots\ldots\ldots\ldots [9]$$

$$2G\eta = 2(1 - \mu)\, \text{Im}\bar{Z} - y\text{Re}Z \ldots\ldots\ldots\ldots [10]$$

For, it is found that these displacements define the stresses in Equations [4] to [6] through Hooke's law, which can be stated [in th]e form

$$\sigma_x = 2G\left[\frac{\partial \xi}{\partial x} + \frac{\mu}{1-2\mu}\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y}\right)\right] \text{ and } \tau_{xy} = G\left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}\right) \ldots [11]$$

A useful observation from Equation [10] is that the value of η at $y = 0$ is

$$\eta_0 = \frac{1-\mu}{G}\text{Im}\bar{Z} = \frac{2(1-\mu^2)}{E}\text{Im}\bar{Z} \ldots\ldots\ldots [12]$$

It is noted, furthermore, that the Airy function defining the stresses by the equations

$$\sigma_x = \frac{\partial^2 F}{\partial y^2},\quad \sigma_y = \frac{\partial^2 F}{\partial x^2},\quad \text{and}\quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \ldots [13]$$

is

$$F = \text{Re}\bar{\bar{Z}} + y\text{Im}\bar{Z} \ldots\ldots\ldots\ldots [14]$$

In a slice parallel to the x, y-plane one unit thick the total vertical force transmitted between two points is the increase of the derivative

$$\frac{\partial F}{\partial x} = \text{Re}\bar{Z} + y\text{Im}Z \ldots\ldots\ldots\ldots [15]$$

between the points. Similarly, the total horizontal force transmitted between two points is the increase of

$$\partial F/\partial y = y\text{Re}Z \ldots\ldots\ldots\ldots [16]$$

between the points.

Introductory Application to Boussinesq's Problem

The semi-infinite solid $y \geqq 0$, with y positive downward, is under consideration. The function

$$Z = P/(i\pi z) \ldots\ldots\ldots\ldots [17]$$

gives

$$\bar{Z} = \frac{P}{i\pi}\log(z/c_1) = \frac{P}{i\pi}\left[\log(r/c) + i\left(\theta - \frac{\pi}{2}\right)\right] \ldots [18]$$

that is

$$\text{Re}\bar{Z} = P[\theta - (\pi/2)]/\pi \ldots\ldots\ldots\ldots [19]$$

According to Equations [15] and [19], in a slice parallel to the xy-plane and one unit thick the total vertical force transmitted between $\theta = \pi$ and $\theta = 0$ is $-P$. It is concluded that Equation [17] represents the solution of Boussinesq's problem in two dimensions for a normal pressure P concentrated at $z = 0$. The familiar formulas for stresses and displacements are obtained readily by substituting from Equations [17] and [18] in Equations [4], [5], [6], [9], and [10].

Rows of Forces

Equation [17] suggests consideration of two modified functions

$$Z_1 = -i\frac{P}{l}\cot(\pi z/l) \quad\text{and}\quad Z_2 = -i\frac{P}{l\sin(\pi z/l)} \ldots [20]$$

Near $z = 0$ both approach Z in Equation [17]. Further inspection shows that Z_1 represents a row of equal pressures P at $z = 0, \pm l, \pm 2l, \ldots$, and Z_2 represents a row of pressures P at $z = 0, \pm 2l, \pm 4l, \ldots$ and a row of pulls P at $z = \pm l, \pm 3l, \pm 5l, \ldots$ on the solid $y \geqq 0$. When y becomes great, Z_1 converges toward $-P/l$, making the stresses in Equations [4] to [6] converge toward a uniform pressure P/l; while Z_2 converges toward zero, making the stresses converge toward zero, as they should under the self-balancing load.

Demonstration by Application to Hertz's Problem of Two Circular Cylinders in Contact

The solid $y \geqq 0$ is considered again. As stress function is chosen

$$Z = -\frac{2P}{\pi a^2}\left[\sqrt{(a^2 - z^2)} + iz\right] \ldots\ldots\ldots [21]$$

or

$$Z = -\frac{2P}{\pi a^2}\left[\sqrt{(a^2 - x^2 + y^2 - i\,2xy)} + ix - y\right] \ldots [22]$$

At $y = 0$ the shearing stress $\tau_{xy} = 0$, and the normal stresses, according to Equations [4] and [5], are both equal to ReZ. Accordingly

$$\sigma_x = \sigma_y = 0 \text{ at } y = 0,\ x < -a \text{ or } x > a \ldots\ldots [23]$$

$$\sigma_x = \sigma_y = -(2P/\pi a^2)\sqrt{(a^2 - x^2)} \quad \text{at } y = 0,\quad -a < x < a \ldots\ldots\ldots [24]$$

Equations [23] and [24] show that the diagram of pressures on the surface $y = 0$ can be drawn as a half-ellipse between $x = -a$ and $x = a$; outside there is no load. The total pressure on the slice one unit thick is P.

When z becomes numerically great, with y remaining positive, one may write

$$\sqrt{(a^2 - z^2)} = -iz\sqrt{[1 - (a^2/z^2)]} = -iz\,(1 - a^2/2z^2 - \ldots) \ldots [25]$$

Therefore, Z in Equation [21] converges toward Z in Equation [17], which represents Boussinesq's problem.

In the interval $-a < x < a$ at $y = 0$ Equations [12] and [22] give

$$\frac{d\eta_0}{dx} = \frac{2(1-\mu^2)}{E}\text{Im}Z = -\frac{4(1-\mu^2)Px}{\pi E a^2} \ldots [26]$$

that is, along the axis of x there is produced a constant concave curvature

$$\frac{1}{R} = -\frac{d^2\eta_0}{dx^2} = \frac{4(1-\mu^2)P}{\pi E a^2} \ldots\ldots\ldots [27]$$

If instead of being initially flat along the axis of x the surface has an initial convex curvature equal to that in Equation [27], under the pressures defined by Equation [24] the surface will be flattened out and become plane in the interval $-a < x < a$; outside this interval it will be flattened out less.

It follows that if two parallel cylinders with radii R are pressed together by the load P per unit of length, the width $2a$ of the strip of contact will be defined by a in Equation [27], which agrees with Hertz's classical solution. With a known, the contact pressures are defined by Equation [24] and the stresses and displacements in the surrounding region by Equations [21], [4], [5], [6], [9], and [10].

Bearing Pressure Through Slightly Waved Surface

Equation [21] suggests investigation of the stress function

$$Z = -\frac{2p\cos(\pi z/l)}{\sin^2(\pi a/l)}\{\sqrt{[\sin^2(\pi a/l) - \sin^2(\pi z/l)]} + i\sin(\pi z/l)\} \quad \ldots [28]$$

as applying to the solid $y \geqq 0$. It is assumed that $a < l/2$. By computing as in Equation [25], it is found that when y is positive and great compared with a, Equation [28] may be replaced by

$$Z = -ip\cot(\pi z/l) \ldots\ldots [29]$$

According to the comments on Equations [20], Z in Equation [29] represents a row of pressures pl with spacing l at $y = 0$, and a uniform pressure p at great values of y.

At the surface $y = 0$ one finds in the interval $-a < x < a$

$$\sigma_x = \sigma_y = \text{Re}Z = -\frac{2p\cos(\pi x/l)}{\sin^2(\pi a/l)}\sqrt{[\sin^2(\pi a/l) - \sin^2(\pi x/l)]} \quad \ldots [30]$$

and in the interval $a < x < l - a$

$$\sigma_x = \sigma_y = \text{Re}Z = 0 \ldots\ldots [31]$$

The function Z is periodic, and the period is l. The values are repeated in the similar intervals. The strips $nl - a < x < nl + a$ are loaded by pressures $-\sigma_y$ defined numerically by Equation [30]; the remaining strips are unloaded.

Within the loaded strips of the surface Equation [28] gives

$$\text{Im}Z = -\frac{p\sin(2\pi x/l)}{\sin^2(\pi a/l)} \ldots\ldots [32]$$

Over the whole surface ImZ is antisymmetrical with respect to the center lines $x = nl/2$ of the strips. By referring to Equation [12] it is then found that within the loaded strips the deflection of the surface can be stated as

$$\eta_0 = \frac{2(1-\mu^2)}{E}\text{Im}\bar{Z} = \frac{(1-\mu^2)pl\,[\cos(2\pi x/l) - 1]}{\pi E\sin^2(\pi a/l)} \quad \ldots [33]$$

with the integration constant being the same for all the loaded strips.

Assume now that instead of being initially flat the surface is slightly waved, having the equation

$$y_0 = \frac{c}{4}[1 - \cos(2\pi x/l)] \ldots\ldots [34]$$

with

$$c = \frac{4(1-\mu^2)pl}{\pi E\sin^2(\pi a/l)} \ldots\ldots [35]$$

Then under the pressures defined by Equation [30] the ordinates $y_0 + \eta_0$ of the deformed surface will be zero within the loaded strips. The loaded strips will be flattened out and be contained in a single plane. A further examination of ImZ as defined by Equation [28] shows that $y_0 + \eta_0$ will be positive between the loaded strips.

It is concluded that if another solid of the same material and shape is placed in contact with the one considered, so that the axis of x becomes an axis of symmetry, and if thereafter a uniform pressure p is produced at numerically large values of y, the contact pressures will be as defined by Equation [30]; the

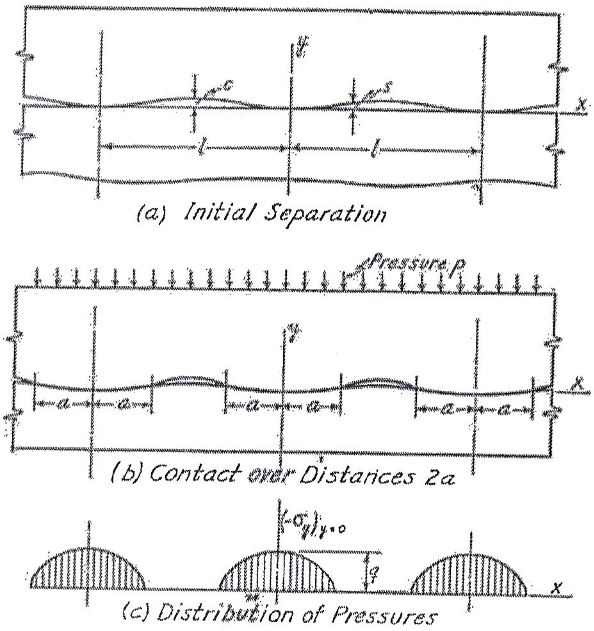

Fig. 1 Bearing Pressures Through Slightly Waved Surface

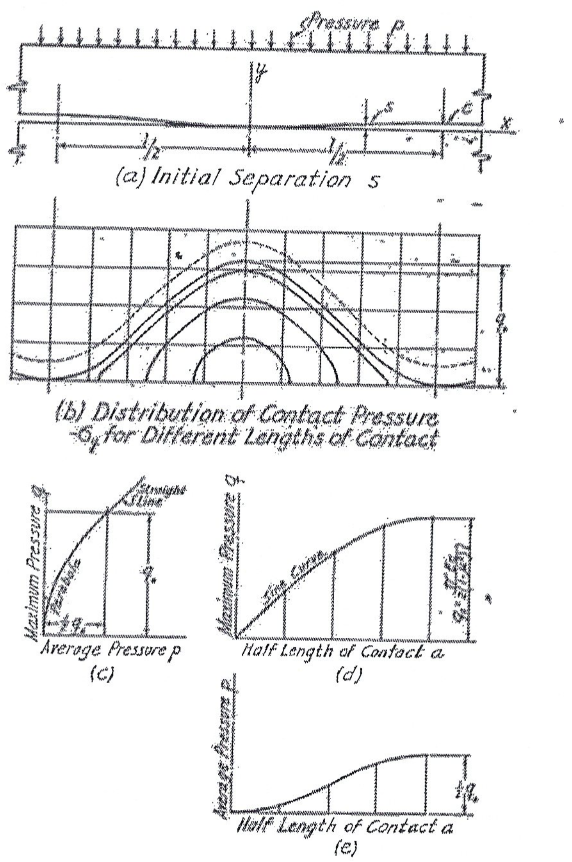

Fig. 2 Diagrams of Bearing Pressures Through Slightly Waved Surface

area of contact through which the pressures are transmitted will consist of the strips of width $2a$ defined by Equation [35]. Equations [34] and [35] are verified by Equation [27] when a is small compared with l.

It is noted that the initial separation of the two surfaces, before pressure is applied, is

$$s = (c/2)[1 - \cos(2\pi x/l)] = c \sin^2(\pi x/l), \quad s_{max} = c \ldots [36]$$

The conclusions that were drawn continue to apply if the two nearly flat surfaces in contact have a different shape, as long as the initial separation is defined by Equations [36].

Fig. 1 illustrates this case. Fig. 2 shows some results obtained from Equations [30] and [35].

Noncircular Cylinder With Nearly Flat Bottom

The function

$$Z = -\frac{8P}{3\pi a^4}\left[(z^2 + a^2/2)\sqrt{(a^2 - z^2)} + iz^3\right] \ldots\ldots [37]$$

applied to the solid $y \geq 0$, is examined first for numerically great values of z. By writing

$$\sqrt{(a^2 - z^2)} = -iz\left(1 - \frac{a^2}{2z^2} - \frac{a^4}{8z^4} - \ldots\right) \ldots\ldots [38]$$

Z in Equation [37] is found to converge toward Z in Equation [17], which represents Boussinesq's problem. Again, at distances great compared with a the stresses are as in Boussinesq's problem, and the total load on the slice one unit wide is P.

At $y = 0$ only the interval $-a < x < a$ is loaded; the pressures are $-\mathrm{Re}Z$. At $x = 0$ the pressure is $8/3\pi$ times the average, that is, less than the average; the maximum pressure occurs at some distance from the center of the load. These pressures can be produced by contact of two solids. The required initial separation s is computed by considering the interval $-a < x < a$. One finds

$$s = -\frac{4(1-\mu^2)}{E}\,\mathrm{Im}\bar{Z} = \frac{8(1-\mu^2)P}{3\pi E a^4}\,x^4 \ldots\ldots [39]$$

The lower solid may have a flat top while the upper solid is a noncircular cylinder shaped at the bottom according to a parabola of fourth degree.

Flat Surfaces With One or More Ridges

Fig. 3(a) shows two solids with surfaces that are initially plane except for a single ridge on one of the surfaces at $x = 0$. Under the pressure p contact is missing in the intervals $-a < x < 0$ and $0 < x < a$. The same situation may be created by driving a plug in between the two surfaces. The stress function

$$Z = -p\sqrt{(1 - a^2/z^2)} \ldots\ldots\ldots\ldots\ldots\ldots [40]$$

represents this case, with the provision that a uniform horizontal tension, for example, $\sigma_x = p$ may be superposed. Fig. 3(b) shows the distribution of the pressures of contact. The force P at the ridge is found by stating Z near $z = 0$ for $y > 0$ in the two forms

$$Z = -ipa/z = P/(i\pi z) \ldots\ldots\ldots\ldots\ldots [41]$$

which gives

$$P = \pi pa \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [42]$$

The value of a will depend not only on p but also on the height and sharpness of the ridge.

Fig. 3(c) shows the related problem of a number of equal parallel ridges with spacing l. The corresponding stress function is

$$Z = -p\sqrt{\left[1 - \frac{\sin^2(\pi a/l)}{\sin^2(\pi z/l)}\right]} \ldots\ldots\ldots\ldots [43]$$

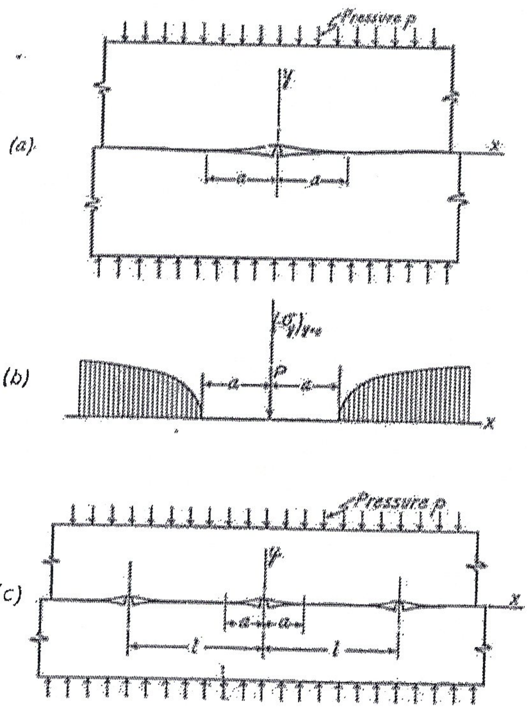

Fig. 3 Contact Between Initially Flat Surfaces With Ridges

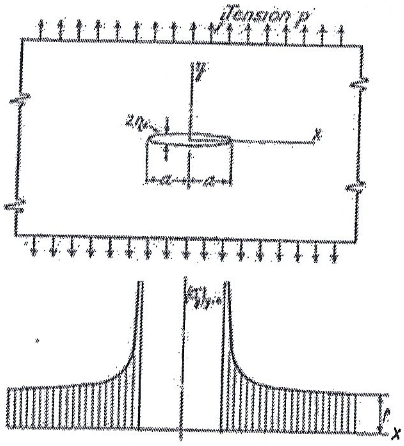

Fig. 4 Internal Crack

Internal Crack

Fig. 4 shows an internal crack which has opened from $z = -a$ to $z = a$ under the influence of an average tension p. The function

$$Z = p/\sqrt{[1 - (a^2/z^2)]} \ldots\ldots\ldots\ldots\ldots [44]$$

solves the problem. Z converges toward p when z becomes numerically great. At $y = 0$ one finds outside the crack the tension

$$\sigma_y = p/\sqrt{[1 - (a^2/x^2)]} \ldots\ldots\ldots\ldots\ldots [45]$$

and within the length of the crack the opening

$$2\eta_0 = \frac{4(1-\mu^2)}{E}\,\mathrm{Im}\bar{Z} = \frac{4(1-\mu^2)p}{E}\sqrt{(a^2 - x^2)} \ldots [46]$$

which shows the shape of the crack to be elliptic. The concentration of stress and the infinite slope $d\eta_0/dx$ at $x = \pm a$ are subject to the usual interpretation applicable to singularities. A uniform horizontal compressive stress p may be superposed without disturbing the remaining features of the solution.

Equation [44] suggests examination of the function

$$Z = p/\sqrt{\left[1 - \frac{\sin^2(\pi a/l)}{\sin^2(\pi z/l)}\right]} \qquad [47]$$

At numerically great values of y this function converges toward p and defines a uniform tension p. At $y = 0$ the function accounts for a system of cracks, each of length $2a$, with centers at $x = 0, \pm l, \pm 2l, \ldots$.

The function

$$Z_1 = Z - p \qquad [48]$$

with Z as in Equation [44] or [47], accounts for a crack or a system of cracks at $y = 0$, created by a liquid pressure p in the cracks as the only load.

Crack Opened by Wedge

Fig. 5(a) shows a crack opened by a wedge exerting pressures P. The stress functions

$$Z_1 = \frac{P}{\pi(a + z)}\sqrt{\frac{a}{z}} \quad \text{and} \quad Z_2 = -\frac{P}{\pi(a + z)}\sqrt{\frac{z}{a}} \qquad [49][50]$$

represent two possible solutions, which require different loads at the outer boundary. Fig. 5(b) and (c), show the corresponding diagrams of stresses at $y = 0$. A change of the load on the outer boundary may bring about the change from Z_1 to Z_2, replacing the concentration of tension in Fig. 5(b) by the diagram of finite compressive stresses in Fig. 5(c). The form of the latter diagram near $x = 0$, with the vertical tangent at $x = 0$, should be considered as characteristic of brittle materials, such as concrete.[6]

[6] "Stresses at a Crack, Size of the Crack, and the Bending of Reinforced Concrete," by H. M. Westergaard, *Journal* American Concrete Inst., November-December, 1933, or, Proceedings, vol. 30, 1934, pp. 93–102. Contains an analysis of this feature of cracks.

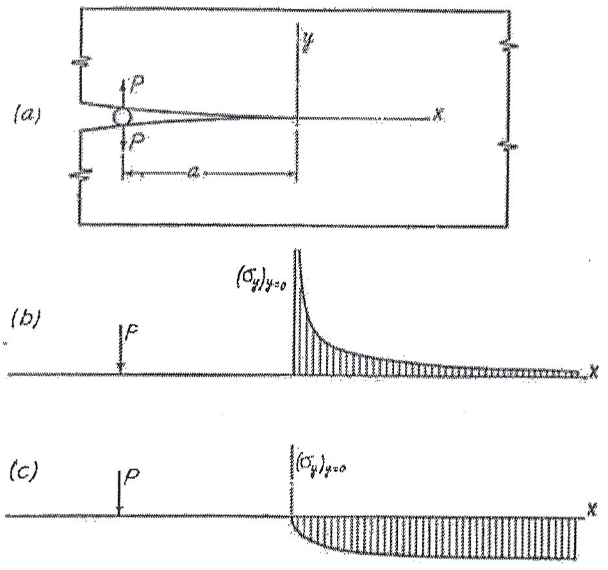

Fig. 5 Crack Opened by Wedge

An internal crack which has been opened between $z = -a$ and $z = a$ by a wedge exerting the pressure P at $z = 0$ is accounted for by the stress function

$$Z = Pa/[\pi z\sqrt{(z^2 - a^2)}] \qquad [51]$$

This function shows concentration of tension at $z = \pm a$, and vanishing stresses at great distances from the crack. If an external pressure is superposed, of the magnitude p defined by Equation [42], Z in Equation [51] will be replaced by Z in Equation [40], and the concentration of tension is replaced by moderate compressive stresses.

Concluding Comment

It is easy to add further examples. Those that have been shown indicate a type of problem to which the method that was used lends itself.